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Controllability of multiple qubit systems*

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Abstract. This paper explores the problem of manipulating multiple-qubit systems when only single-qubit operations or two-qubit-interactive operations are permitted. It is demonstrated that if there exist 2 directional control Hamiltonian for each individual qubit, and one interactive Hamiltonian for each pair of qubits, then multiple qubit systems are open-loop controllable. An important observation of physical interest is emphasized: when only single-qubit operations or two-qubit-interactive operations are permitted, only $n(n+3)/2$ control Hamilton may guarantee open-loop controllability of n qubit systems, and $n(n+3)$ is, in the restricted sense, also the lower limit on the number of operators needed for controllability. At last, we demonstrate that an n-quantum-dot system is open-loop controllable even when only single-qubit operations or two-qubit-interactive operations are permitted.

PACS. 03.65.-w Quantum mechanics – 02.30.Yy Control theory – 02.20.Sv Lie algebras of Lie groups

1 Introduction

Quantum mechanical control theory is being considered as an essential step in the way from quantum physics to quantum technology [1] and has been developed ever since 1980s [2–4]. The conditions for controllability of quantum systems have been discussed by many researchers [2,5–16]. With the development of quantum information processing, the link between controllability and quantum computation has also been explored [17]. To authors' knowledge, the controllability condition of multiple qubit systems when only single-qubit operations or two-qubit-interactive operations are permitted, however, has not been explored yet. This paper investigates this problem and obtains the corresponding controllability conditions of multiple qubit systems.

To discuss the Lie algebraic conditions of controllability of multiple qubit systems, we further reveal some properties of $su(2^n)$. From the view point of physical operation, we study how to generate a particular orthogonal basis of $su(2^n)$, the vectors of which are all tensor product of identity and Pauli matrices. Two important findings of physical interest are emphasized and revealed in this paper: (1) the Lie algebraic $su(2^n)$ can be generated by only $n(n+3)/2$ traceless skew-Hermitian matrices; (2) the special orthogonal basis of $su(2^n)$ may be generated by at least $n(n+3)/2$ vectors in the basis. Based on these observation, we concretely demonstrate controllability conditions of multiple-qubit systems.

It should be mentioned that $n(n+3)/2$ is not the lower limit on the number of operators needed for controllability of n-qubit systems in general. In fact, it has been recognized in 1970s [18] that two matrices generate the full Lie algebra. However, when only single-qubit operations or two-qubit-interactive operations are permitted, $n(n+3)/2$ is the lower limit on the number of operators needed for controllability of n-qubit systems.

The paper is organized as follows. In Section 2, we review the open-loop controllability notation of quantum systems. In Sections 3 and 4, we demonstrate some special properties of $su(2^n)$ and obtain controllability conditions of multiple-qubit systems when only single-qubit operations or two-qubit-interactive operations are permitted. Furthermore, we demonstrate from the view point of physical operation that n-quantum-dot systems are controllable. At last, the paper concludes with some comments.

2 Open-loop controllability of quantum systems

Before discussing controllability of multiple qubit systems, we first briefly review the concept of open-loop controllability of multilevel quantum mechanics systems. As for multilevel quantum mechanical systems, four different notions of controllability of physical interest have already

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been defined and discussed [12]: Operator Controllability (OC), Pure-State Controllability (PSC), Equivalent State Controllability (ESC) and Density Matrix Controllability (DMC). It has been demonstrated that the two notions of OC (in the special unitary case) and DMC are equivalent and they are the strongest among the four controllability notions, whereas PSC and ESC are themselves equivalent. Therefore, we just recite the definitions of Pure-State Controllability (PSC) and Density Matrix Controllability (DMC)

Consider a multilevel quantum dynamical system described by a finite dimensional bilinear model

$$|\dot{\psi}(t)\rangle = \left(A + \sum_{i=1}^{m} u_i(t) B_i\right) |\psi(t)\rangle \tag{1}$$

where $|\psi(t)\rangle$ is the state vector varying on the complex sphere S_C^{N-1}, defined as the set of N-ples of complex numbers $x_j + iy_j; j = 1, \ldots, n$, with $\sum_{j=1}^{N}(x_j^2 + y_j^2) = 1$. The matrices $A, B_1, \ldots, B_m$ are in the Lie algebra $su(N)$ of skew-Hermitian matrices with zero trace of N dimension.

The solution of equation (1) at time t, $|\psi(t)\rangle$ with initial condition $|\psi_0\rangle$, is given by $|\psi(t)\rangle = X(t)|\psi(t)\rangle$ where $X(t)$ is the solution at time t of

$$\dot{X}(t) = \left(A + \sum_{i=1}^{m} u_i(t) B_i\right) X(t) \tag{2}$$

with initial condition $X(0) = I$.

Definition 1. The system (1) is pure-state controllable if, for every pair of initial and final states $|\psi_0\rangle$ and $|\psi_1\rangle$ in S_C^{N-1}, there exist control functions $u_1, u_2, \ldots, u_m$ and a time $t > 0$ such that the solution of (1) at time t, with initial condition $|\psi_0\rangle$, is $|\psi(t)\rangle = |\psi_1\rangle$.

Definition 2. The system (1) is density matrix controllable if, for each pair of unitarily equivalent density matrices ρ_1 and ρ_2, there exist control functions $u_1, u_2, \ldots, u_m$ and a time $t > 0$ such that the solution of (2) at time t, $X(t)$, satisfies $X(t)\rho_1 X^*(t) = \rho_2$.

Denote $\boldsymbol{L}$ the Lie algebra generated by $\{A, B_1, B_2, \ldots, B_m\}$. Then we have the following necessary and sufficient conditions for density matrix controllability.

Lemma 1 [7]. *The system (1) is density matrix controllable if and only if* $\mathbf{L} = su(N)$ *or* $\mathbf{L} = u(N)$.

Lemma 2 [7]. *The system (1) is pure state controllable if and only if* $\mathbf{L}$ *is isomorphic (conjugate) to* $sp(n/2)$ *or to* $su(n)$*, for* N *even, or to* $su(N)$*, for* N *odd.*

3 Property of $su(2^n)$

It is well-known that the unitary evolution of n interacting spin $\frac{1}{2}$ particles is described by an element of $SU(2^n)$, the special unitary group of dimension 2^n. The corresponding Lie algebraic $su(2^n)$ is a $4^n - 1$ dimensional space.

First recall that the Pauli spin matrices $\sigma_x, \sigma_y, \sigma_z$ defined by

$$\sigma_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3}$$

$$\sigma_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{4}$$

$$\sigma_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5}$$

are the generators of the rotation in the two-dimensional Hilbert space and a basis for the Lie algebra of traceless skew-Hermitian matrices $su(2)$. They obey the well-known relations

$$[\sigma_x, \sigma_y] = i\sigma_z, \quad [\sigma_y, \sigma_z] = i\sigma_x, \quad [\sigma_z, \sigma_x] = i\sigma_y. \tag{6}$$

Obviously, one can choose an orthogonal basis $\{iB_s\}$ for the Lie algebra $su(2^n)$ taking the form [19,20]

$$B_s = 2^{q-1}\prod_{k=1}^{n}(\sigma_{k\alpha})^{a_{ks}} \tag{7}$$

with $\alpha = x, y$, or z and

$$\sigma_{k\alpha} = I_2 \bigotimes \ldots \bigotimes \sigma_\alpha \bigotimes \ldots I_2 \tag{8}$$

where q is an integer taking values between 1 and n, σ_α, the Pauli matrix appears in the above equation (8) only at the kth position, and I_2, the two-dimensional identity matrix appears everywhere except at the kth position. a_{ks} is 1 in q of the indices and 0 in the remaining.

Denote the Lie algebra generated by $\{i\sigma_{jx}, i\sigma_{jy}, 2i\sigma_{lz}\sigma_{mz}; j = 1, 2, \ldots, n; 1 \le l < m \le n\}$ as $L_{(n)}^z$. We can demonstrate the following Lemma.

Lemma 3. $L_{(n)}^z = su(2^n)$.

Proof. Firstly, we can demonstrate that for any j and $\alpha_j = x, y, z$, $i\sigma_{j\alpha_j}$ is in the $L_{(n)}^z$. This is true because, for any j, $i\sigma_{jx}$ and $i\sigma_{jx}$ are in $L_{(n)}^z$, and the following equation is satisfied

$$[\sigma_{jx}, \sigma_{jy}] = i\sigma_{jz}. \tag{9}$$

Secondly, we can show that $2i\sigma_{l\alpha_l}\sigma_{m\alpha_m}$ is in the $L_{(n)}^z$ for any l, m and any $\alpha_l, \alpha_m = x, y$, or z. This conclusion can be made from the fact that all $2i\sigma_{lz}\sigma_{mz}$ with $j = 1, 2, \ldots, n; 1 \le l < m \le n$ are in $L_{(n)}^z$ and the following equations are satisfied:

$$[\sigma_{ly}, \sigma_{lz}\sigma_{mz}] = i\sigma_{lx}\sigma_{mz} \tag{10}$$

$$[\sigma_{my}, \sigma_{lz}\sigma_{mz}] = i\sigma_{lz}\sigma_{mx} \tag{11}$$

$$[\sigma_{ly}, \sigma_{lz}\sigma_{mx}] = i\sigma_{lx}\sigma_{mx} \tag{12}$$

$$[\sigma_{lz}\sigma_{mx}, \sigma_{lx}] = i\sigma_{ly}\sigma_{mx} \tag{13}$$

$$[\sigma_{lz}\sigma_{mz}, \sigma_{lx}] = i\sigma_{ly}\sigma_{mz} \tag{14}$$

$$[\sigma_{lz}\sigma_{mz}, \sigma_{mx}] = i\sigma_{lz}\sigma_{my} \tag{15}$$

$$[\sigma_{ly}\sigma_{mz}, \sigma_{mx}] = i\sigma_{ly}\sigma_{my} \tag{16}$$

$$[\sigma_{ly}\sigma_{my}, \sigma_{lz}] = i\sigma_{lx}\sigma_{my}. \tag{17}$$

Thirdly, we can demonstrate that if for $k \geq 2$, all iB_s with $2 \leq q \leq k$ are in the $L^z_{(n)}$, then all iB_s with $q = k+1$ are in the $L^z_{(n)}$.

In fact, since there exists the following recursive equation

$$[\sigma_{j_1\alpha_{j_1}}\sigma_{j_2\alpha_{j_2}}\cdots\sigma_{j_k\beta_{j_k}}, \sigma_{j_k\gamma_{j_k}}\sigma_{j_{k+1}\alpha_{j_{k+1}}}] \tag{18}$$

$$= \sigma_{j_1\alpha_{j_1}}\sigma_{j_2\alpha_{j_2}}\cdots[\sigma_{j_k\beta_{j_k}}, \sigma_{j_k\gamma_{j_k}}]\sigma_{j_{k+1}\alpha_{j_{k+1}}} \tag{19}$$

for all $\alpha_{j_1}, \alpha_{j_2}, \cdots, \alpha_{j_{k-1}}, \gamma_{j_k}, \sigma_{j_k}, \beta_{j_k}, \alpha_{j_{k+1}} = x, y$, or z and

$$\pm i\sigma_{j_k\alpha_{j_k}} = [\sigma_{j_k\beta_{j_k}}, \sigma_{j_k\gamma_{j_k}}] \tag{20}$$

holds for all possible $\alpha_{j_k} = x, y$,or z, we can conclude that all possible iB_s with $q = k+1$ are in the $L^z_{(n)}$.

Therefore, all iB_s with $q = 1, 2, \ldots, n$ must be in the $L^z_{(n)}$. This completes the proof.

Remark. 1. Denote the Lie algebra generated by $\{i\sigma_{jx}, i\sigma_{jz}, 2i\sigma_{ly}\sigma_{my}; j = 1, 2, \ldots, n; 1 \leq l < m \leq n\}$ as $L^y_{(n)}$, and denote the Lie algebra generated by $\{i\sigma_{jy}, i\sigma_{jz}, 2i\sigma_{lx}\sigma_{mx}; j = 1, 2, \ldots, n; 1 \leq l < m \leq n\}$ as $L^x_{(n)}$. We can also demonstrate that $L^y_{(n)} = su(2^n)$ and $L^x_{(n)} = su(2^n)$.

2. Since $2i\sigma_{lz}\sigma_{mz}$ can be generated by $\{i\sigma_{lx}, i\sigma_{ly}, i\sigma_{mx}, i\sigma_{my}, 2i(\sigma_{lx}\sigma_{mx} + \sigma_{ly}\sigma_{my} + \sigma_{lz}\sigma_{mz})\}$ for $1 \leq l < m \leq n$, we can also conclude from Lemma 3 that the Lie algebra generated by $\{i\sigma_{jx}, i\sigma_{jy}, 2i(\sigma_{lx}\sigma_{mx} + \sigma_{ly}\sigma_{my} + \sigma_{lz}\sigma_{mz}); j = 1, 2, \ldots, n; 1 \leq l < m \leq n\}$ is $su(2^n)$.

Further more, we can have the following theorem:

Theorem 1. *Let $\alpha, \beta, \gamma = x, y$,or z, and $\alpha \neq \beta$, then* (1) *The Lie algebra generated by $\{i\sigma_{j\alpha}, i\sigma_{j\beta}, 2i\sigma_{l\gamma}\sigma_{m\gamma}; j = 1, 2, \ldots, n; 1 \leq l < m \leq n\}$ is $su(2^n)$.* (2) *The Lie algebra generated by $\{i\sigma_{j\alpha}, i\sigma_{j\beta}, 2i(\sigma_{lx}\sigma_{mx}+\sigma_{ly}\sigma_{my}+\sigma_{lz}\sigma_{mz}); j = 1, 2, \ldots, n; 1 \leq l < m \leq n\}$ is $su(2^n)$.*

Remark. This theorem not only demonstrates that the Lie algebra $su(2^n)$ can be generated by $n(n+3)/2$ traceless skew-Hermitian matrices, but also implies that if there exist 2 directional control Hamiltonian for each individual qubit, and one interactive Hamiltonian for each pair of qubits, then multiple qubit systems are density matrix controllable, and subsequently are pure state controllable.

Another important fact should be pointed out:

Theorem 2. *An orthogonal basis $\{iB_s\}$ of the Lie algebra $su(2^n)$ can be generated by at least $n(n+3)/2$ tensor product vectors in the orthogonal basis $\{iB_s\}$.*

Proof. First, we demonstrate that the theorem is true for $n = 1, 2$. When $n = 1$, it is obvious that an orthogonal basis $\{iB_s\}$ of the Lie algebra $su(2)$ can be generated by at least 2 tensor product vector in the orthogonal basis.

When $n = 2$, it is easy to demonstrate that an orthogonal basis $\{iB_s\}$ of the Lie algebra $su(4)$ can be generated by at least 5 tensor product vectors in the orthogonal basis.

Second, we demonstrate that if the theorem is true for $n = k$ then the theorem is true for $n = k+1$.

Suppose that an orthogonal basis $\{iB_s\}$ of the Lie algebra $su(2^k)$ can be generated by at least $k(k+3)/2$ tensor product vectors in the orthogonal basis $\{iB_s\}$. With $k(k+3)/2$ tensor product vectors in the orthogonal basis $\{iB_s\}$ in hand, one need at least additional $k+2$ tensor product vectors to generate an orthogonal basis $\{iB_s\}$ of the Lie algebra $su(2^{k+1})$: 2 vectors from $\{i\sigma_{(k+1)x}, i\sigma_{(k+1)y}, i\sigma_{(k+1)z}\}$ and k tensor product vectors of the form $\{i\sigma_{(k+1)\alpha_j}\sigma_{j\beta_j}, j = 1, 2, ..., k\}$ with $\alpha_j, \beta_j = x, y$ or z. That is to say, an orthogonal basis $\{iB_s\}$ of the Lie algebra $su(2^{k+1})$ can be generated by at least $(k+1)(k+4)/2$ tensor product vectors in the orthogonal basis $\{iB_s\}$.

So far, we can conclude that the theorem is true for any $n \geq 1$. This completes the proof.

Remark. For many physical systems, the adjustable control Hamilton is of the form $i\sigma_{i\alpha_i}$ or $\{i\sigma_{j\beta_j}\sigma_{k\gamma_k}\}$ with $i, j, k \in N$ and $\alpha_i, \beta_j, \gamma_k = x, y$ or z. From Theorem 1 and 2, we can conclude that when only single-qubit operations or two-qubit-interactive operations are permitted, $n(n+3)/2$ is the lower limit on the number of operators needed for controllability of n-qubit systems. In fact, from the view point of mathematics, one can easily demonstrate that at least $n(n+3)/2$ control Hamilton are needed for controllability of n-qubit systems when only single-qubit operations or two-qubit-interactive operations are permitted. One can make this conclusion from the fact that there should exist two single-qubit operations for each qubit and one interactive operation for each pair of qubits, i.e., $n(n-1)/2 + 2n = n(n+3)/2$.

4 Open loop controllability of multiple qubit systems

Now we can demonstrate that some multiple-qubit systems are open loop controllable.

Result 1. One qubit system given by

$$|\dot{\psi}(t)\rangle = -i\left(u_x(t)\sigma_x + u_y(t)\sigma_y\right)|\psi(t)\rangle \tag{21}$$

is density matrix controllable (and is also pure state controllable).

Result 2. A two-coupled-qubit system given by

$$|\dot{\psi}(t)\rangle = -i\left(H_d + \sum_{i=1}^{4} u_i(t)H_i\right)|\psi(t)\rangle \tag{22}$$

where

$$H_d = 2\pi\lambda\sigma_z \bigotimes \sigma_z \tag{23}$$

$$H_1 = 2\pi\sigma_x \bigotimes I_2 \tag{24}$$

$$H_2 = 2\pi\sigma_y \bigotimes I_2 \tag{25}$$

$$H_3 = 2\pi I_2 \bigotimes \sigma_x \tag{26}$$

$$H_4 = 2\pi I_2 \bigotimes \sigma_y \tag{27}$$

is density matrix controllable (and is also pure state controllable).

Remark. It should be mentioned that equation (22) can be used to describe the system of two heteronuclear spins.

Similarly, we can obtain the following result about open loop controllability conditions of n qubit systems.
Result 3. An n-qubit system given by

$$|\dot{\psi}(t)\rangle = -i\left(\sum_{l=1}^{n-1}\sum_{m=l+1}^{n} H_{lm}(t) + \sum_{i=1}^{n}(H_{ix}(t)+H_{iy}(t))\right)|\psi(t)\rangle \quad (28)$$

with

$$H_{lm}(t) = \lambda_{lm}(t)\sigma_{lz}\sigma_{mz} \quad (29)$$
$$H_{ix}(t) = u_{ix}(t)\sigma_{ix} \quad (30)$$
$$H_{iy}(t) = u_{iy}(t)\sigma_{iy} \quad (31)$$

is density matrix controllable (and is also pure state controllable).

As another concrete physical example, we will show that an n quantum-dot system is open-loop controllable.

Quantum dots are fabricated from semiconductor materials, metals, or small molecules [21,22]. They work by confining electric charge quanta (i.e., spins) in three dimensional boxes with electrostatic potentials. The spin of a charge quantum in a single quantum dot can be manipulated, i.e., single qubit operations, by applying pulsed local electromagnetic fields, through a scanning-probe tip, for example. Two-qubit operations can be achieved by spectroscopic manipulation or by a purely electrical gating of the tunneling barrier between neighboring quantum dots. Usually, an n-quantum-dot system can be described by Hubbbard model [23]

$$i\hbar|\dot{\psi}(t)\rangle = \left(\sum_{1\le l<m\le n} H_{lm}(t) + \sum_{j=1}^{n}(H_{jx}(t)+H_{jy}(t)+H_{jz}(t))\right)|\psi(t)\rangle \quad (32)$$

where

$$H_{lm}(t) = 4J_{lm}(t)(\sigma_{lx}\sigma_{mx}+\sigma_{ly}\sigma_{my}+\sigma_{lz}\sigma_{mz}) \quad (33)$$
$$H_{jx}(t) = 2\mu_B g_j(t)b_{jx}(t)\sigma_{jx} \quad (34)$$
$$H_{jy}(t) = 2\mu_B g_j(t)b_{jy}(t)\sigma_{jy} \quad (35)$$
$$H_{jz}(t) = 2\mu_B g_j(t)b_{jz}(t)\sigma_{jz}. \quad (36)$$

It is easy to conclude that the system (32) is pure state controllable. In this case, $n(n+5)/2$ control Hamilton are performed on the n-quantum-dot system. However, from the view point of physical operation, we demonstrate that n-quantum-dot systems are controllable even when only single-qubit operations or two-qubit-interactive operations are permitted.

5 Conclusions and discussions

In this paper, we explore the controllability condition of multiple qubit systems when only single-qubit operations or two-qubit-interactive operations are permitted. It has been found that when only single-qubit operations or two-qubit-interactive operations are permitted, only $n(n+3)/2$ control Hamilton may guarantee open-loop controllability of n qubit systems, and $n(n+3)$ is also the lower limit on the number of operators needed for controllability in the restricted sense. This is in contrast to the fact that only 2 control Hamilton may guarantee open-loop controllability of N-level quantum systems in some situations (see for example [13]).

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